On a Closed Binding Curve of One-holed Torus

Weiqiang Wu
Department of Mathematics
University of Maryland, College Park
waikng@math.umd.edu

October 18, 2011

Abstract

Given a closed binding curve γ of a surface Σ, any equivalence class of marked complete hyperbolic structure can be decomposed into polygons(possibly with a puncture) with sides being hyperbolic geodesic segments. When Σ is a one-holed torus and $\gamma = A^3B^2$, we show that any equivalence class of marked complete hyperbolic structure gives rise to an equilateral bigon with a puncture and a hexagon with equal opposite sides. In particular, we give a new coordinates of the Fricke Space of the one-holed torus.

1 Introduction

The length function of a simple closed curve on the Teichmüller Space $\mathcal{T}_\Sigma$ (Fricke Space $\mathcal{F}_\Sigma$) of a closed surface Σ has been studied by S.Kerckhoff [Ke80] and S.Wolpert [Wo87] since 1980's, and it plays an important role in understanding the geometry of $\mathcal{T}_\Sigma$. One of the most important features of these length functions is that they are convex along *Earthquake Paths* (or *Weil-Petersson geodesics*) [Ke80, Wo87].

Definition 1.1 (Fricke Space)**.** $\mathcal{F}_\Sigma := \left\{(f, X) \mid \Sigma \xrightarrow{f} X \textit{ is a diffeomorphism}\right\} / \sim$, *where X is a complete hyperbolic surface and $(f, X) \sim (g, Y)$ if there exists a hyperbolic isometry $i : X \to Y$ such that the following diagram commutes up to isotopy.*

$$\begin{tikzcd} \Sigma \arrow[r, "f"] \arrow[d, equal] & X \arrow[d, "i"'] \\ \Sigma \arrow[r, "g"] & Y \end{tikzcd}$$

Definition 1.2 (Closed Binding Curves)**.** *Let Σ be a surface (possibly with punctures), a closed curve γ is binding if γ intersects itself in a minimal position and $\Sigma - \gamma$ is a union of disjoint disks (possibly with a puncture).*

Let γ be a closed binding curve of Σ, the length function $l_\gamma : \mathcal{F}_\Sigma \to \mathbb{R}$ is not only *strictly convex* along *Earthquake Paths* (or *Weil-Petersson geodesics*) but also *proper*. Consequently, it has a *unique* minimum at some marked hyperbolic structure $[f_\gamma, X_\gamma] \in \mathcal{F}_\Sigma$.

Also, for any $[f, X] \in \mathcal{F}_\Sigma$, the unique closed hyperbolic geodesic isotopic to $f(\gamma)$ will cut X into polygons(possibly with a puncture) with sides being hyperbolic geodesics, and conversely given these polygons $[f, X]$ can be reconstructed. Therefore these polygons may provide invariants and combinatoric ways to study $\mathcal{F}_\Sigma$.

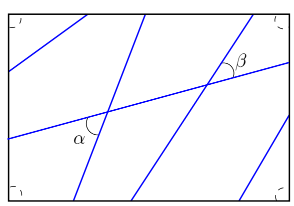


Figure 1: The closed binding curve A^3B^2 in the one-holed torus.

When $\Sigma_{1,1}$ is the one-holed torus and $\gamma = A^3B^2$ as shown in Figure 1, any $[f, X] \in \mathcal{F}_{\Sigma_{1,1}}$ can be decomposed into

- A bigon with side lengths a and b and angles α and β, which forms a cusp region.

- A Hexagon with side lengths a, c, d, b, c' and d' and angles $\pi-\beta$, α, $\pi-\beta$, $\pi-\alpha$, β and $\pi-\alpha$ such that $c=c'$ and $d=d'$.

as in Figure 2.

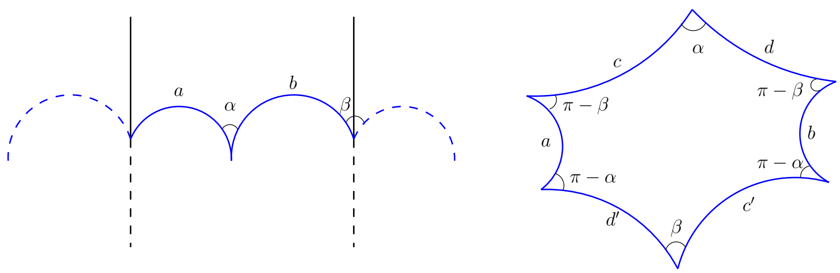


Figure 2: The punctured bigon and hexagon in a decomposition.

Let $\mathcal{P} := \{$ compatible pairs of a punctured bigon and a hexagon$\}$, then there is a 1-1 correspondence

$$\mathcal{P} \underset{c}{\overset{g}{\rightleftharpoons}} \mathcal{F}_{\Sigma_{1,1}}$$

which comes from gluing and cutting. Let $\mathcal{P}_0 \subset \mathcal{P}$ be the subset consisting of those with $\alpha=\beta$ and $a=b$.

Definition 1.3 (Length Function on $\mathcal{P}$)**.** *The length function $l : \mathcal{P} \to \mathbb{R}$ is given by $a+b+c+d$.*

Therefore the diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, "g"] \arrow[d, "l"'] & \mathcal{F}_{\Sigma_{1,1}} \arrow[d, "l_\gamma"'] \\ \mathbb{R} \arrow[r, equal] & \mathbb{R} \end{tikzcd}$$

commutes by construction of l.

2 Main Theorems

Theorem 2.1. $\mathcal{P}_0 = \mathcal{P}$*. In particular, for any* $[f,X] \in \mathcal{F}_{\Sigma_{1,1}}$*,*

$$\alpha = \beta \in \left(0, \frac{2\pi}{3}\right),$$

and

$$a = b = \log\left(\frac{1+\cos\left(\frac{\alpha}{2}\right)}{1-\cos\left(\frac{\alpha}{2}\right)}\right).$$

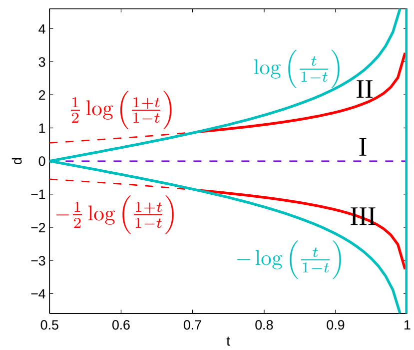


Figure 3: The region of $\mathcal{V}$.

First, we need the following lemmas.

Lemma 2.1 (Existence)**.** *There exists an injective map $j : \mathcal{V} \to \mathcal{P}_0$, where*

$$\mathcal{V} := \left\{(t, s) \in \mathbb{R}^2 \mid -\log\left(\frac{t}{1-t}\right) < s < \log\left(\frac{t}{1-t}\right), \frac{1}{2} < t < 1\right\}.$$

This lemma will be proved in Section 3.

Remark

- s will be the displacement of the mid point of a from the common perpendicular of a and b.
- t will be $\cos\left(\frac{\alpha}{2}\right)$.

Lemma 2.2 (Properness)**.** *The pullback of the length function $j^*(l) : \mathcal{V} \to \mathbb{R}$ is proper.*

This lemma will be proved in Section 4.

Theorem 2.2. *$g \circ j : \mathcal{V} \to \mathcal{F}_{\Sigma_{1,1}}$ is a diffeomorphism.*

Proof of Theorem 2.2. Note that $\mathcal{F}_{\Sigma_{1,1}}$ is diffeomorphic to $\mathbb{R}^2$, hence $g \circ j$ is an injective local diffeomorphism.

In addition $g \circ j$ is proper, since $j^*(l)$ and l_γ are proper by Lemma 2.2 and the following diagram commutes.

$$\begin{tikzcd} \mathcal{V} \arrow[r, "g\circ j"] \arrow[d, "j^*(l)"'] & \mathcal{F}_{\Sigma_{1,1}} \arrow[d, "l_\gamma"'] \\ \mathbb{R} \arrow[r, equal] & \mathbb{R} \end{tikzcd}$$

Therefore $g \circ j$ is a diffeomorphism by *Invariance of Domain* (See [Ha]). $\square$

Proof of Theorem 2.1. $g \circ j$ is onto by Theorem 2.2, hence $P_0 = P$. Since $\cos\left(\frac{\alpha}{2}\right) = t \in \left(\frac{1}{2}, 1\right)$ (see remark following lemma 2.1),

$$\alpha = \beta \in \left(0, \frac{2\pi}{3}\right)$$

and

$$a = b = \log\left(\frac{1+\cos\left(\frac{\alpha}{2}\right)}{1-\cos\left(\frac{\alpha}{2}\right)}\right)$$

follows from the following lemma. □

Lemma 2.3. *If $a = b$ and $\alpha = \beta \in (0, \pi)$, then*

$$a = \log\left(\frac{1+\cos\left(\frac{\alpha}{2}\right)}{1-\cos\left(\frac{\alpha}{2}\right)}\right).$$

Proof. Use the upper half plane model with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The length of the geodesic segment a is given by

$$a = \int_{\frac{\alpha}{2}}^{\pi - \frac{\alpha}{2}} \frac{d\theta}{\sin(\theta)} = \log\left(\frac{1+\cos\left(\frac{\alpha}{2}\right)}{1-\cos\left(\frac{\alpha}{2}\right)}\right).$$

□

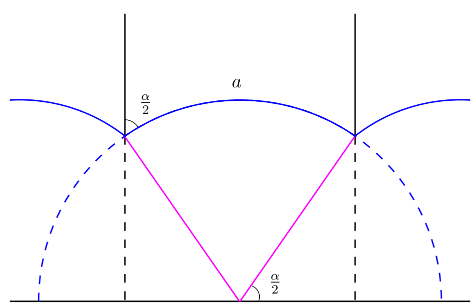


Figure 4: Relation between a and α in the cusp.

3 Existence of Compatible Pairs from $\mathcal{V}$

In this section, we will prove **Lemma 2.1**, i.e.

There exists an injective map $j : \mathcal{V} \to \mathcal{P}_0$, where

$$\mathcal{V} := \left\{ (t,s) \in \mathbb{R}^2 \mid -\log\left(\frac{t}{1-t}\right) < s < \log\left(\frac{t}{1-t}\right), \frac{1}{2} < t < 1 \right\}. \tag{1}$$

Given $t \in \left(\frac{1}{2}, 1\right)$, Lemma 2.3 also guarantees the existence of the punctured bigon with

$$\begin{aligned} \alpha = \beta &= 2\cos^{-1}(t) \\ a = b &= \log\left(\frac{1+t}{1-t}\right). \end{aligned} \tag{2}$$

We divide $\mathcal{V}$ into three parts as also shown in Figure 3.

- $I = \left\{ (t,s) \in \mathcal{V} \mid -\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) < s < \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) \right\}$
- $II = \left\{ (t,s) \in \mathcal{V} \mid s < -\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) \text{ or } s > \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) \right\}$
- $III = \left\{ (t,s) \in \mathcal{V} \mid s = \pm\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) \right\}$

Injectivity of j will follow from our construction.

3.1 Existence of Type I Hexagons

For any (t,s) in

$$I = \left\{ (t,s) \in \mathcal{V} \mid -\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) < s < \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) \right\} \tag{3}$$

we are going to construct the hexagon as shown in Figure 5.

- First of all, α and a is determined by (2) and let

$$\begin{aligned} a_1 &:= \frac{a}{2} + s = \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) + s \\ a_2 &:= \frac{a}{2} - s = \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) - s. \end{aligned} \tag{4}$$

  Therefore by (1) and (3),

$$a_1, a_2 \in \begin{cases} \left(\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{t}{1-t}\right), \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) + \log\left(\frac{t}{1-t}\right)\right) & , \quad t \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right) \\ \left(0, \log\left(\frac{1+t}{1-t}\right)\right) & , \quad t \in \left[\frac{1}{2}, 1\right) \end{cases} \tag{5}$$

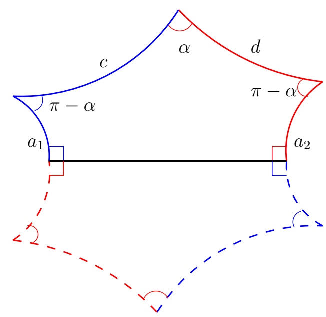


Figure 5: Shape of a Type I Hexagon.

- Second, we need to show

  Proposition 3.1. *The geodesic ray along c does not intersect the common perpendicular on the right; and the geodesic ray along d does not intersect the common perpendicular on the left either. (see Figure 5)*

  Recall the following fact from hyperbolic geometry (See [Ra]).

  Lemma 3.1. *The area of a hyperbolic triangle (possibly with ideal vertices) is given by*

  $$\pi - \alpha - \beta - \gamma$$

  where α, β, γ are the inner angles. In particular, $\alpha + \beta + \gamma < \pi$.

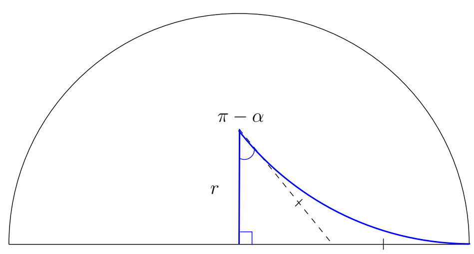


Figure 6: Non-intersecting Condition.

Proof of Proposition 3.1. When $t \in \left[\frac{\sqrt{2}}{2}, 1\right)$ (i.e. $\alpha \in \left(0, \frac{\pi}{2}\right]$), the proposition is obviously true from Lemma 3.1. Therefore we may assume $t \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$ (i.e. $\alpha \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$).

From Figure 6, the Euclidean length of r, which is the least distance to keep away from intersecting, is given by

$$\frac{\cos(\pi - \alpha)}{1 + \sin(\pi - \alpha)} = -\frac{\cos(\alpha)}{1 + \sin(\alpha)}.$$

Therefore using the Poincaré disk model with the hyperbolic metric

$$ds^2 = 4\frac{dx^2 + dy^2}{\left(1 - (x^2 + y^2)\right)^2},$$

$$\begin{aligned}
r &= \int_0^{-\frac{\cos(\alpha)}{1+\sin(\alpha)}} \frac{2dr}{1-r^2} \\
&= \log\left(\frac{1+\sin(\alpha)-\cos(\alpha)}{1+\sin(\alpha)+\cos(\alpha)}\right) \\
&= \log\left(\frac{1+2t\sqrt{1-t^2}-(2t^2-1)}{1+2t\sqrt{1-t^2}+(2t^2-1)}\right) \\
&= \frac{1}{2}\log(1-t) + \frac{1}{2}\log(1+t) - \log(t). \qquad (6)
\end{aligned}$$

and hence $a_1 > r$ and $a_2 > r$ from (5). Therefore the proposition is true as well. $\square$

- And last, we are ready to show the existence of Type I hexagons.

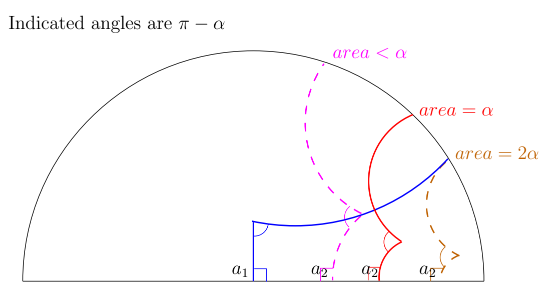


Figure 7: Existence of Type I Hexagons.

Proof. Without loss of generality, we may assume $s \leq 0$, then $a_1 \leq a_2$.

See Figure 7. Since the *blue line* does not intersect the *horizonal line* by Proposition 3.1 and $a_1 \leq a_2$, there is a *magenta line* such that its vertex is on the *blue line*. They form a quadrilateral with angles $\pi - \alpha$, $\frac{\pi}{2}$, $\frac{\pi}{2}$ and some nonzero angle, therefore its area is less than α. (When $s = 0$, it is a degenerate quadrilateral with area $0 < \alpha$)

Continue parallel translating the *magenta line* to the right, then there is a *brown line* such that it intersects the *blue line* at infinity. They form a pentagon with angles $\pi - \alpha$, $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\pi - \alpha$ and 0, therefore its area is equal to 2α.

Therefore there is a unique *red line* between the *magenta line* and the *brown line*, such that the area bounded by it together with the *blue line* is exactly α, hence the angle between the *blue line* and the *red line* is α.

By doubling the pentagon formed by the *blue line* and the *red line*, we find the desired Type I hexagon. $\square$

3.2 Existence of Type II Hexagons

For any (t, s) in

$$II = \left\{ (t, s) \in \mathcal{V} \mid s < -\frac{1}{2} \log \left(\frac{1+t}{1-t} \right) \text{ or } s > \frac{1}{2} \log \left(\frac{1+t}{1-t} \right) \right\}. \tag{7}$$

From (1) and (7), $t \in \left(\frac{\sqrt{2}}{2}, 1 \right)$ (i.e. $\alpha \in \left(0, \frac{\pi}{2}\right)$. And without loss of generality, we assume $s < 0$ in the following.

We are going to construct the hexagon as shown in Figure 8.

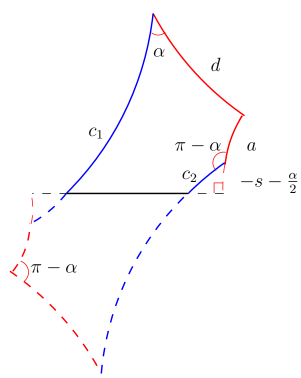


Figure 8: Shape of a Type II Hexagon.

- First of all, we need to construct the right triangle at the right corner of Figure 8.

Proposition 3.2. *There exists a right triangle with a side of length $-s-\frac{a}{2}$ and the other adjacent angle being α.*

Proof. Since

$$-\log \left(\frac{t}{1-t} \right) < s < -\frac{1}{2} \log \left(\frac{1+t}{1-t} \right)$$

from (1) and (7) and

$$a = \log \left(\frac{1+t}{1-t} \right),$$

from (2),

$$-s - \frac{a}{2} \in \left(0, \log\left(\frac{t}{1-t}\right) - \frac{1}{2}\log\left(\frac{1+t}{1-t}\right)\right). \tag{8}$$

Use the same picture as in Figure 6 but replacing the angle $\pi - \alpha$ with α, the least distance from being intersecting is given by

$$\begin{aligned} r' &= \log\left(\frac{1+\sin(\alpha)+\cos(\alpha)}{1+\sin(\alpha)-\cos(\alpha)}\right) \\ &= \log\left(\frac{1+2t\sqrt{1-t^2}+(2t^2-1)}{1+2t\sqrt{1-t^2}-(2t^2-1)}\right) \\ &= \log(t) - \frac{1}{2}\log(1-t) - \frac{1}{2}\log(1+t). \end{aligned} \tag{9}$$

Then $-s - \frac{a}{2} < r'$ by (8), hence there exists a unique such right triangle. $\square$

- Note in Figure 8 that c_1 is parallel to c_2 along the common perpendicular and the geodesic along d does not intersect the common perpendicular on the left since $\pi - \alpha > \frac{\pi}{2}$. Then we are ready to show the existence of Type II hexagons.

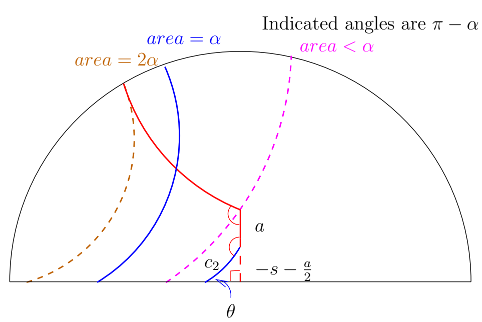


Figure 9: Existence of Type II Hexagons.

Proof. See Figure 9. Parallel translating c_2 along the *horizontal line* to the left, there is a *magenta line* such that it meets the *red line* at the vertex, together with c_2 and the *horizontal line* they form a quadrilateral with angles $\theta, \pi - \theta, \pi - \alpha$ and some nonzero angle, therefore its area is less than α.

Continue parallel translating the *magenta line* to the left, then there is a *brown line* such that it intersects the *red line* at infinity. They form a

pentagon with angles $\theta, \pi-\theta, \pi-\alpha, \pi-\alpha$ and 0, therefore its area is equal to 2α.

Therefore there is a unique *blue line* between the *magenta line* and the *brown line*, such that the area inscribed by it together with the *red line* and c_2 is exactly α, hence the angle between the *blue line* and the *red line* is α.

By doubling the pentagon formed by the *blue line*, the *red line* and c_2, we find the desired Type II hexagon. $\square$

3.3 Existence of Type III Hexagons

For any (t, s) in

$$III = \left\{ (t,s) \in \mathcal{V} \mid s = \pm \frac{1}{2} \log \left(\frac{1+t}{1-t} \right) \right\}. \tag{10}$$

From (1) and (7), $t \in \left(\frac{\sqrt{2}}{2}, 1\right)$ (i.e. $\alpha \in \left(0, \frac{\pi}{2}\right)$. And without loss of generality, we assume $s = -\frac{1}{2} \log \left(\frac{1+t}{1-t}\right)$ in the following.

We are going to construct the hexagon as shown in Figure 10.

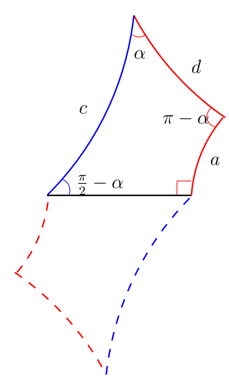


Figure 10: Shape of a Type III Hexagon.

- First of all, like Proposition 3.1 the geodesic ray along d does not intersect the common perpendicular on the left either in this case, since $\pi - \alpha > \frac{\pi}{2}$.

- Then we are ready to show the existence of Type III hexagons.

  Proof. See Figure 11. There is a *magenta line* such that it meets the *red line* at the vertex, they form a triangle with angles $\frac{\pi}{2} - \alpha, \frac{\pi}{2}$ and some nonzero angle, therefore its area is less than α.

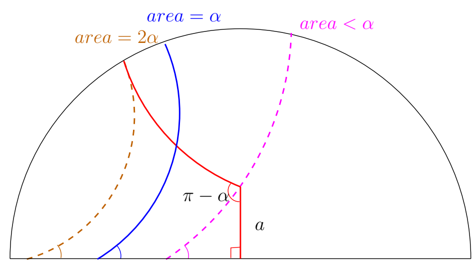


Figure 11: Existence of Type III Hexagons.

Continue parallel translating the *magenta line* to the left, then there is a *brown line* such that it intersects the *red line* at infinity. They form a quadrilateral with angles $\frac{\pi}{2} - \alpha$, $\frac{\pi}{2}$, $\pi - \alpha$ and 0, therefore its area is equal to 2α.

Therefore there is a unique *blue line* between the *magenta line* and the *brown line*, such that the area inscribed by it together with the *red line* is exactly α, hence the angle between the *blue line* and the *red line* is α.

By doubling the quadrilateral formed by the *blue line* and the *red line*, we find the desired Type III hexagon. □

This concludes the proof of Lemma 2.1.

4 Properness of the Length Function

In this section, we will prove **Lemma 2.2**, i.e.

The pullback of the length function $j^*(l) : \mathcal{V} \to \mathbb{R}$ is proper, where l is given by $a + b + c + d$.

If suffices to show that if any sequence $\{(t_n, s_n)\} \subset \mathcal{V}$ leaves any compact set of $\mathcal{V}$, then $j^*(l)\left((t_n, s_n)\right) \to \infty$.

We may assume $\{t_n\}$ converges to $\hat{t} \in \left[\frac{1}{2}, 1\right]$. Then there are four cases to consider

- $\hat{t} = 1$
- $\hat{t} \in \left(\frac{\sqrt{2}}{2}, 1\right)$
- $\hat{t} \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$
- $\hat{t} = \frac{\sqrt{2}}{2}$

4.1 Case: $\hat{t} = 1$

Proof. Since $t_n \to \hat{t} = 1$, from (2)

$$a_n = \log\left(\frac{1+t_n}{1-t_n}\right) \to \infty,$$

and note that $j^*(l)\left((t_n, s_n)\right) > a_n$ hence

$$j^*(l)\left((t_n, s_n)\right) \to \infty.$$

$\square$

4.2 Case: $\hat{t} \in \left(\frac{\sqrt{2}}{2}, 1\right)$

Proof. In this case, without loss of generality we may assume $s_n < 0$, $\{(t_n, s_n)\} \subset II$ and $\{s_n\}$ converges to $\hat{s} = -\log\left(\frac{\hat{t}}{1-\hat{t}}\right)$.

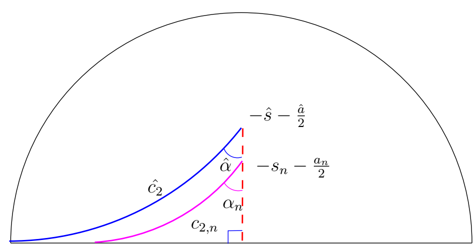


Figure 12: $t_n \to \hat{t} \in \left(\frac{\sqrt{2}}{2}, 1\right)$.

We consider again the right triangle as in Figure 8. We lift all the geodesic segments $-s_n - \frac{a_n}{2}$'s on the *vertical line* at the origin as in Figure 12. Since

$$-\hat{s} - \frac{\hat{a}}{2} = \log\left(\frac{\hat{t}}{1-\hat{t}}\right) - \frac{1}{2}\log\left(\frac{1+\hat{t}}{1-\hat{t}}\right) = r' > 0$$

from (9). The geodesic along $\hat{c_2}$ is intersecting the *horizontal line* at infinity, hence $\hat{c_2} = \infty$.

Since $(t_n, s_n) \to (\hat{t}, \hat{s})$,

$$\alpha_n \to \hat{\alpha}$$
$$-s_n - \frac{a_n}{2} \to -\hat{s} - \frac{\hat{a}}{2}$$

then

$$c_{2,n} \to \hat{c_2} = \infty.$$

Since $j^*(l)\left((t_n, s_n)\right) > c_n > c_{2,n}$,

$$j^*(l)\left((t_n, s_n)\right) \to \infty.$$

$\square$

4.3 Case: $\hat{t} \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$

Proof. In this case, without loss of generality we may assume $s_n \leq 0$, $\{(t_n, s_n)\} \subset I$ and $\{s_n\}$ converges to $\hat{s} = -\log\left(\frac{\hat{t}}{1-\hat{t}}\right)$.

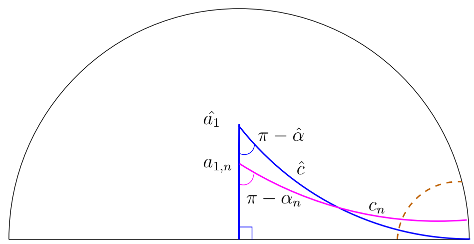


Figure 13: $t_n \to \hat{t} \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

We lift all the geodesic segments $a_{1,n}$'s on the *vertical line* at the origin as in Figure 13. Since

$$\hat{a_1} = \frac{\hat{a}}{2} + \hat{s} = \frac{1}{2}\log\left(\frac{1+\hat{t}}{1-\hat{t}}\right) - \log\left(\frac{\hat{t}}{1-\hat{t}}\right) = r > 0$$

from (5) and (6), the geodesic along $\hat{c}$ intersects the *horizontal line* at infinity, hence $\hat{c} = \infty$.

For any $N > 0$, we can choose a *brown line* geodesic, which is perpendicular to the *horizontal line*, such that the distance between the *vertical line* and the *brown line* is greater than N.

Since $(t_n, s_n) \to (\hat{t}, \hat{s})$,

$$\alpha_n \to \hat{\alpha}$$

$$a_{1,n} \to \hat{a_1}.$$

Therefore the geodesic along c_n has to intersect the *brown line*, when n is large enough.

Note that for any Type II hexagon, c and a_2 are parallel (See Figure 5). Therefore the geodesic along $a_{2,n}$ has to be on the right hand side of the *brown line*. Therefore $a_{1,n} + c_n + d_n + a_{2,n} > N$ by triangular inequality. Let $N \to \infty$,

$$a_{1,n} + c_n + d_n + a_{2,n} \to \infty.$$

Since $j^*(l)\left((t_n, s_n)\right) > a_n + c_n + d_n = a_{1,n} + c_n + d_n + a_{2,n}$,

$$j^*(l)\left((t_n, s_n)\right) \to \infty.$$

□

4.4 Case: $\hat{t} = \frac{\sqrt{2}}{2}$

Proof. In this case, without loss of generality we may assume $s_n < 0$, $\{(t_n, s_n)\}$ is either completely contained in I, II or III and $\{s_n\}$ converges to

$$\hat{s} = -\log\left(\frac{\hat{t}}{1-\hat{t}}\right) = -\log\left(\sqrt{2}+1\right).$$

From (2)

$$\hat{\alpha} = 2\cos^{-1}(\hat{t}) = \frac{\pi}{2}$$

$$\hat{a} = \log\left(\frac{1+\hat{t}}{1-\hat{t}}\right) = 2\log\left(\sqrt{2}+1\right).$$

- If $\{(t_n, s_n)\} \subset I$, from (5)

$$\hat{a_1} = \frac{\hat{a}}{2} + \hat{s} = 0.$$

Therefore

$$\alpha_n \to \frac{\pi}{2}$$

$$a_{1,n} \to 0.$$

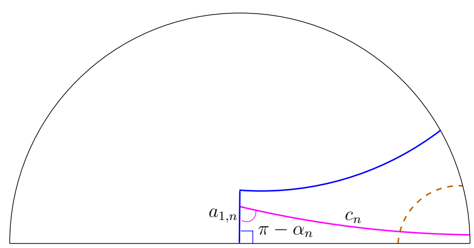


Figure 14: $\alpha_n \to \frac{\pi}{2}$ and $a_{1,n} \to 0$.

We lift all the geodesic segments $a_{1,n}$'s on the *vertical line* at the origin as in Figure 14. For any $N > 0$, we can choose a *brown line* geodesic, which is perpendicular to the *horizontal line*, such that the distance between the *vertical line* and the *brown line* is greater than N.

Then when n is large enough, the geodesic along c_n has to intersect the *brown line*. Since the area bounded by c_n and the *brown line* approaches to 0, the geodesic along $a_{2,n}$ has to be on the right hand side of the *brown line* when n is even larger enough. Therefore $a_{1,n} + c_n + d_n + a_{2,n} > N$ by triangular inequality. Let $N \to \infty$,

$$a_{1,n} + c_n + d_n + a_{2,n} \to \infty.$$

Since $j^*(l)\left((t_n, s_n)\right) > a_n + c_n + d_n = a_{1,n} + c_n + d_n + a_{2,n}$,

$$j^*(l)\left((t_n, s_n)\right) \to \infty.$$

- If $\{(t_n, s_n)\} \subset II$,

  $$-\hat{s} - \frac{\hat{a}}{2} = 0.$$

  Therefore

  $$\alpha_n \to \frac{\pi}{2}$$

  $$a_n + (-s_n, \frac{a_n}{2}) \to 2\log\left(\sqrt{2}+1\right).$$

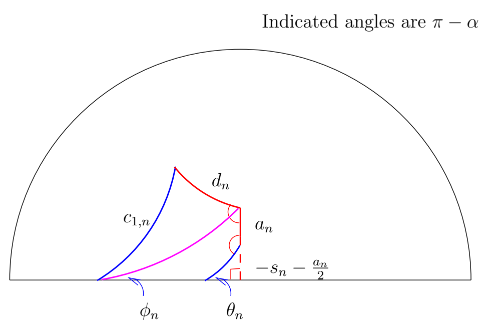
  

  Figure 15: $\phi_n \to 0$ and $a_n + (-s_n, \frac{a_n}{2}) \to 2\log\left(\sqrt{2}+1\right)$.

  We lift all the geodesic segments $-s_n - \frac{a_n}{2}$'s on the *vertical line* at the origin as in Figure 15. Since $\phi_n < \theta_n < \pi - \alpha_n - \frac{\pi}{2}$,

  $$\phi_n \to 0.$$

  Consider the right triangle bounded by the *magenta line*, the *red line* and the *horizontal line*. The length of the *magenta line* approaches to infinity by the *Law of Sine*, hence

  $$c_{1,n} + d_n \to \infty$$

  by triangular inequality. Since $j^*(l)\left((t_n, s_n)\right) > c_n + d_n > c_{1,n} + d_n$,

  $$j^*(l)\left((t_n, s_n)\right) \to \infty.$$

- If $\{(t_n, s_n)\} \subset III$, See Figure 16. The same argument above can be applied to this case as well.

$\square$

This concludes the properness of $j^*(l)$.

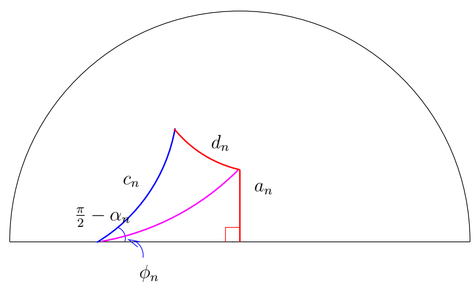


Figure 16: $\phi_n \to 0$ and $a_n \to 2\log\left(\sqrt{2}+1\right)$.

5 The Hyperbolic Structure with Minimal Length

Let

$$\mathcal{A} := \left\{(t,0) \mid t \in \left(\frac{1}{2}, 1\right)\right\} \subset \mathcal{V}.$$

Since $j^*(l)$ has a unique minimum and it is even on s, the minimum is obtained at some $(t_0, 0) \in \mathcal{A}$.

In fact, $j^*(l)\mid_{\mathcal{A}}$ is quite explicit.

Theorem 5.1. *$j^*(l)\mid_{\mathcal{A}}$: $\left(\frac{1}{2}, 1\right) \to \mathbb{R}$ is given by*

$$t \mapsto 2\log\left(\frac{\sqrt{5t^2-1}+2t^2}{(2t-1)(1-t)}\right).$$

Corollary 5.1. *$j^*(l)$ has a unique minimal at $t_0 = \frac{3\sqrt{5}}{10}$. Hence the hyperbolic structure with the minimal γ length is given by the hexagon with*

$$\alpha = \beta = 2\cos^{-1}\left(\frac{3\sqrt{5}}{10}\right)$$

$$a = b = \log\left(\frac{29+12\sqrt{5}}{11}\right)$$

$$c = d = \log\left(\frac{21+8\sqrt{5}}{11}\right).$$

We need the following Lemmas from [Ra].

Lemma 5.1. *Let Q be a hyperbolic convex quadrilateral with two adjacent right angles, opposite angles α, β, and sides of length c, d between α, β and the right angles, respectively. Then*

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta)+\cosh(d)}{\sin(\alpha)\sin(\beta)}.$$

Lemma 5.2. *Let Q be a hyperbolic convex quadrilateral with three right angles and fourth angle γ, and let a, b the lengths of sides opposite the angle γ. Then*

$$\cos(\gamma) = \sinh(a)\sinh(b).$$

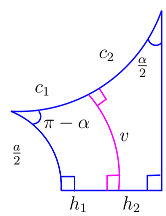


Figure 17: A quarter of a hexagon for $t \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

Proof of Theorem 5.1. We first want to find the formula for $t \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$. From Figure 17 and use the above lemmas,

$$\begin{aligned}
\cosh(v) &= \cosh\left(\frac{a}{2}\right)\sin(\pi - \alpha) \\
\cosh(h_1) &= \cosh(c_1)\sin(\pi - \alpha) \\
\cos(\pi - \alpha) &= \sinh(h_1)\sinh(v) \\
\cosh(h_2) &= \cosh(c_2)\sin\left(\frac{\alpha}{2}\right) \\
\cos\left(\frac{\alpha}{2}\right) &= \sinh(h_2)\sinh(v).
\end{aligned}$$

Then

$$\begin{aligned}
\cos^2(\alpha) &= \left(\cosh^2(c_1)\sin^2(\alpha) - 1\right)\left(\cosh^2\left(\frac{a}{2}\right)\sin^2(\alpha) - 1\right) \\
\cos^2\left(\frac{\alpha}{2}\right) &= \left(\cosh^2(c_2)\sin^2\left(\frac{\alpha}{2}\right) - 1\right)\left(\cosh^2\left(\frac{a}{2}\right)\sin^2(\alpha) - 1\right).
\end{aligned} \tag{11}$$

Use (2)

$$\begin{aligned}
\cosh^2(c_1) &= \frac{t^2}{(4t^2-1)(1-t^2)} \\
\cosh^2(c_2) &= \frac{5t^2-1}{(4t^2-1)(1-t^2)}.
\end{aligned} \tag{12}$$

Hence

$$\begin{aligned}
c_1 &= \log\left(\frac{(2t+1)(1-t)}{\sqrt{(4t^2-1)(1-t^2)}}\right) \\
c_2 &= \log\left(\frac{\sqrt{5t^2-1} + 2t^2}{\sqrt{(4t^2-1)(1-t^2)}}\right).
\end{aligned} \tag{13}$$

Then

$$c = c_1 + c_2 = \log\left(\frac{\sqrt{5t^2-1}+2t^2}{(2t-1)(1+t)}\right). \tag{14}$$

It can be shown that (14) works for all $t \in \left(\frac{1}{2}, 1\right)$. Therefore

$$j^*(l) = a + b + c + d = 2a + 2c = 2\log\left(\frac{\sqrt{5t^2-1}+2t^2}{(2t-1)(1-t)}\right). \tag{15}$$

□

References

[FM] Benson Farb and Dan Margalit. A primer on mapping class groups.

[Ha] Allen Hatcher. Algebraic topology. Cambridge University Press, 2002.

[Ra] John G. Ratcliff. Foundations of hyperbolic manifolds, 2nd ed. Graduate texts in mathematics, 2006.

[EE] Clifford J. Earle and James Eells. A fibre bundle description of Teichmüller theory. J. Differential Geometry, 3:19C43, 1969.

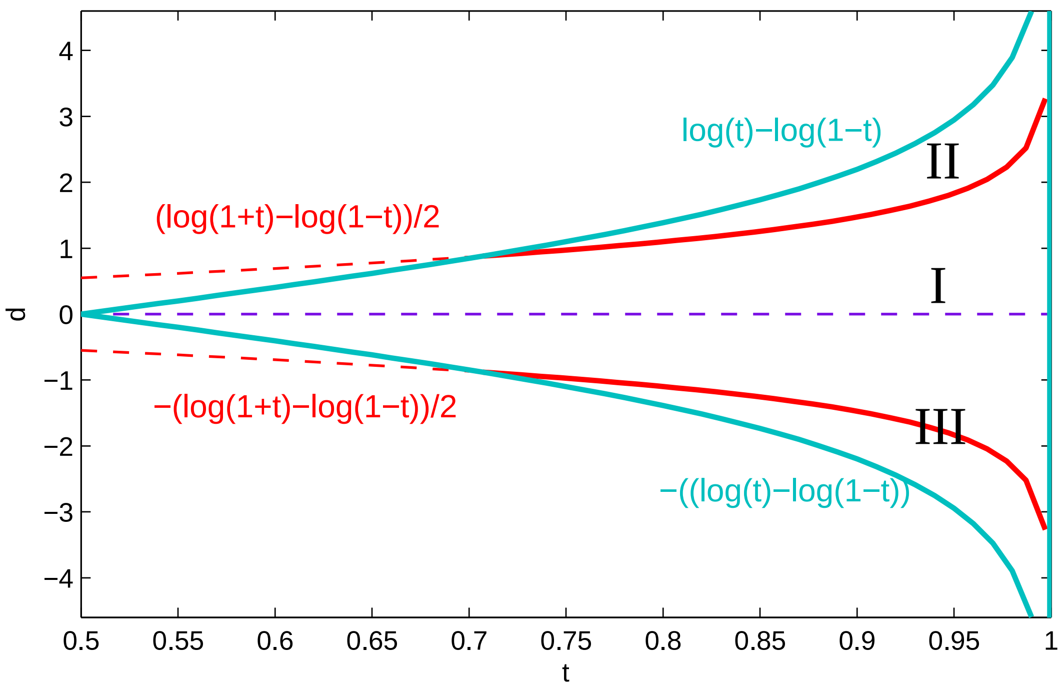
[FLP] A. Fathi, F. Laudenbach, and V. Poénaru, Travaux de Thurston sur les surfaces, Astérisque 66-67 (1979).

[Go86] William M. Goldman. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. Invent. Math. 85 (1986), no. 2, 263–302.

[Ke80] Steven P. Kerckhoff. The Nielsen realization problem. Bull. Amer. Math. Soc. (N.S.) 2 (1980), no. 3, 452–454.

[Wo83] Scott A. Wolpert. On the symplectic geometry of deformations of a hyperbolic surface. Ann. of Math. (2) 117 (1983), no. 2, 207–234.

[Wo87] Scott A. Wolpert. Geodesic length functions and the Nielsen problem. J. Differential Geom. 25 (1987), no. 2, 275–296.


log(t)−log(1−t)
II
(log(1+t)−log(1−t))/2
I
−(log(1+t)−log(1−t))/2
III
−((log(t)−log(1−t))
d
t
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
4
3
2
1
0
−1
−2
−3
−4